Il metodo
delle differenze finite

MODI IN STRUTTURE OMOGENEE

note per il corso di
"Metodi numerici per l'elettromagnetismo"

a cura di M. Politi - Politecnico di Milano - a.a. 2004/05

Generalità

I modi in una struttura guidante omogenea sono la soluzione dell'equazione scalare di Helmholtz:

$$\nabla^2\Phi = -k_c^{\,2}\Phi, \qquad \text{con } k_c^{\,2} = \gamma^2 + k^2$$

dove:

$$\gamma = \pm\sqrt{k_c^{\,2} - k^2} \quad \rightarrow \quad \text{costante di propagazione}$$
$$k = \omega\sqrt{\mu\varepsilon} \quad \rightarrow \quad \text{numero d'onda}$$

Si dimostra che la soluzione esiste per un insieme discreto di k_c (autovalori), dipendenti dalle condizioni al contorno.

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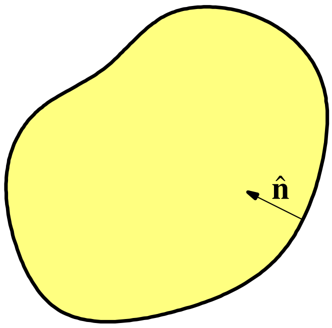
Modi per $k_c \neq 0$



Condizioni al contorno:

	modi TE ($\Phi \equiv H_z$)	modi TM ($\Phi \equiv E_z$)
parete elettrica	$\partial\Phi/\partial n=0$	$\Phi=0$
parete magnetica	$\Phi=0$	$\partial\Phi/\partial n=0$

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Impostazione integrale

Integrando l'equazione di Helmholtz su un volume generico, si trova:

$$\int_V \nabla^2 \Phi \, dV = -k_c^2 \int_V \Phi \, dV$$

$$\int_S \nabla \Phi \, d\vec{S} =$$

Le equazioni alle differenze possono essere ricavate da questa facendo riferimento alle areole elementari del reticolo cartesiano.

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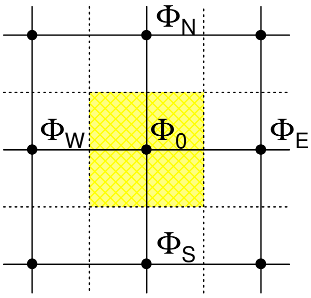
Eq. alle differenze (caso generale)



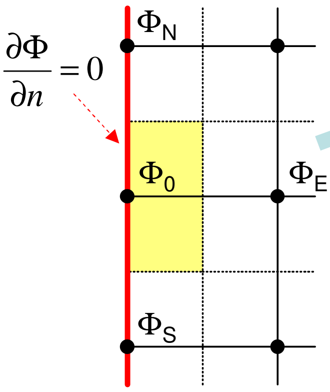
Per l’areola evidenziata si può scrivere:

$$\int_{S_0} \nabla\Phi \, d\vec{S} \cong (\Phi_N - \Phi_0) + (\Phi_S - \Phi_0) + (\Phi_E - \Phi_0) + (\Phi_W - \Phi_0) \cong \Phi_N + \Phi_S + \Phi_E + \Phi_W - 4\Phi_0$$

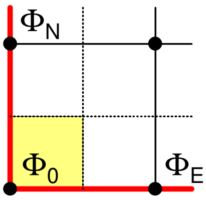
$$\int_{V_0} \Phi \, dV \cong \Phi_0 h^2$$

$$\Phi_N + \Phi_S + \Phi_E + \Phi_W - 4\Phi_0 = -k_c^2 h^2 \Phi_0$$

Eq. alle differenze (contorno)

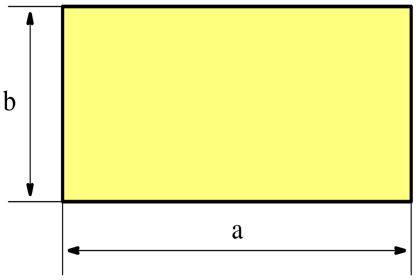


$$\frac{1}{2}\Phi_N + \frac{1}{2}\Phi_S + \Phi_E - 2\Phi_0 = -k_c^2 \frac{h^2}{2}\Phi_0$$



$$\frac{1}{2}\Phi_N + \frac{1}{2}\Phi_E - \Phi_0 = -k_c^2 \frac{h^2}{4}\Phi_0$$

Esempio: guida rettangolare



$$b = \frac{a}{2}$$

Modi TE

TE_{10} $\quad \lambda_c = 2a$ ⟸ modo dominante

TE_{20} $\quad \lambda_c = a$

TE_{01} $\quad \lambda_c = a$ ⟸ modi degeneri

TE_{11} $\quad \lambda_c = \frac{2a}{\sqrt{5}}$

Modi TM

TM_{11} $\quad \lambda_c = \frac{2a}{\sqrt{5}}$

TM_{21} $\quad \lambda_c = \frac{a}{\sqrt{2}}$

TM_{12} $\quad \lambda_c = \frac{2a}{\sqrt{17}}$

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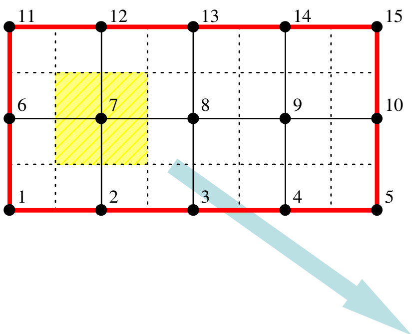
Modi TE: sistema di equazioni



$$
\begin{cases}
\frac{1}{2}\Phi_2 + \frac{1}{2}\Phi_6 - \Phi_1 = -k_c^2 \frac{h^2}{4}\Phi_1 \\
\frac{1}{2}\Phi_1 + \frac{1}{2}\Phi_3 + \Phi_7 - 2\Phi_2 = -k_c^2 \frac{h^2}{2}\Phi_2 \\
\frac{1}{2}\Phi_2 + \frac{1}{2}\Phi_4 + \Phi_8 - 2\Phi_3 = -k_c^2 \frac{h^2}{2}\Phi_3 \\
\frac{1}{2}\Phi_3 + \frac{1}{2}\Phi_5 + \Phi_9 - 2\Phi_4 = -k_c^2 \frac{h^2}{2}\Phi_4 \\
\frac{1}{2}\Phi_4 + \frac{1}{2}\Phi_{10} - \Phi_5 = -k_c^2 \frac{h^2}{4}\Phi_5 \\
\frac{1}{2}\Phi_1 + \Phi_7 + \frac{1}{2}\Phi_{11} - 2\Phi_6 = -k_c^2 \frac{h^2}{2}\Phi_6 \\
\Phi_2 + \Phi_6 + \Phi_8 + \Phi_{12} - 4\Phi_7 = -k_c^2 h^2 \Phi_7 \\
\dots \\
\frac{1}{2}\Phi_{10} + \frac{1}{2}\Phi_{14} - \Phi_{15} = -k_c^2 \frac{h^2}{4}\Phi_{15}
\end{cases}
$$

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Modi TE: equazione matriciale

$$\begin{bmatrix}
-1 & 1/2 & \circ & \circ & \circ & 1/2 & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\
1/2 & -2 & 1/2 & \circ & \circ & 1 & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\
\circ & 1/2 & -2 & 1/2 & \circ & \circ & \circ & 1 & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\
\circ & \circ & 1/2 & -2 & 1/2 & \circ & \circ & \circ & 1 & \circ & \circ & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & 1/2 & -1 & \circ & \circ & \circ & \circ & 1/2 & \circ & \circ & \circ & \circ & \circ \\
1/2 & \circ & \circ & \circ & \circ & -2 & 1 & \circ & \circ & \circ & 1/2 & \circ & \circ & \circ & \circ \\
\circ & 1 & \circ & \circ & \circ & 1 & -4 & 1 & \circ & \circ & \circ & 1 & \circ & \circ & \circ \\
 & & & & & & & \cdots & & & & & & & \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & 1/2 & \circ & \circ & \circ & 1/2 & -1
\end{bmatrix}
\begin{bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \\ \Phi_4 \\ \Phi_5 \\ \Phi_6 \\ \Phi_7 \\ \vdots \\ \Phi_{15} \end{bmatrix}
= -k_c^2 h^2 \,\mathrm{diag}
\begin{bmatrix} 1/4 \\ 1/2 \\ 1/2 \\ 1/2 \\ 1/4 \\ 1/2 \\ 1 \\ \vdots \\ 1/4 \end{bmatrix}
\begin{bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \\ \Phi_4 \\ \Phi_5 \\ \Phi_6 \\ \Phi_7 \\ \vdots \\ \Phi_{15} \end{bmatrix}$$

$$\mathbf{A\Phi} = \lambda \mathbf{B\Phi}$$

problema agli autovalori generalizzato

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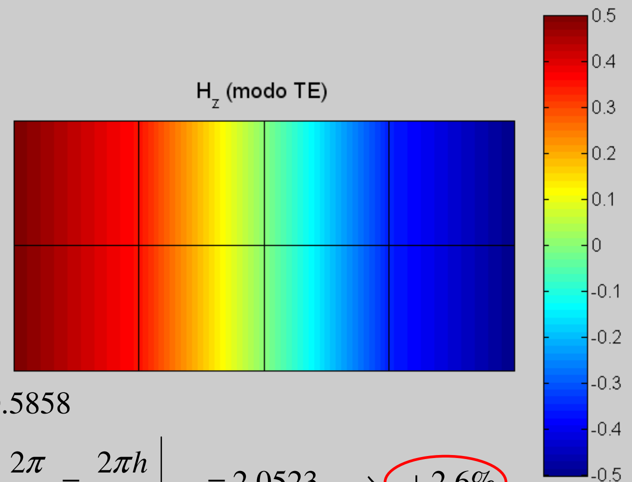
1° modo TE (dominante)



$$\lambda_1 = 0.5858$$

$$\frac{\tilde{\lambda}_{c1}}{a} = \frac{2\pi}{a\tilde{k}_{c1}} = \left.\frac{2\pi h}{a\sqrt{\lambda_1}}\right|_{\frac{a}{h}=4} = 2.0523 \;\rightarrow\; +2.6\%$$

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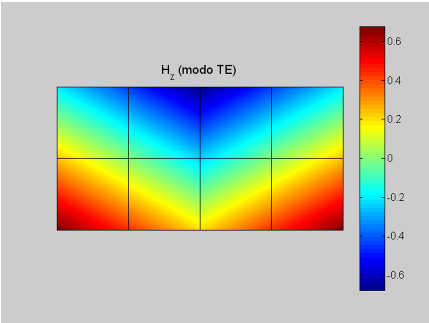
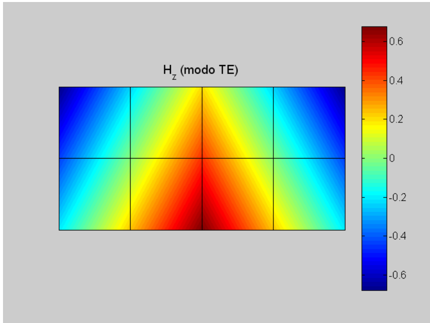
2° e 3° modo TE (degeneri)





$$\lambda_2 = 2.0000$$

$$\frac{\tilde{\lambda}_{c2}}{a} = \frac{2\pi}{a\tilde{k}_{c2}} = \left.\frac{2\pi h}{a\sqrt{\lambda_2}}\right|_{\frac{a}{h}=4} = 1.1107 \quad \rightarrow \quad +11\%$$

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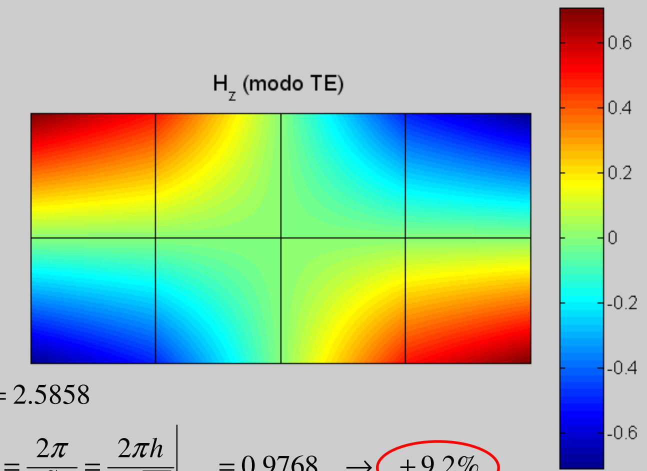
4° modo TE



$$\lambda_4 = 2.5858$$

$$\frac{\tilde{\lambda}_{c4}}{a} = \frac{2\pi}{a\tilde{k}_{c4}} = \left.\frac{2\pi h}{a\sqrt{\lambda_4}}\right|_{\frac{a}{h}=4} = 0.9768 \quad \rightarrow \quad +9.2\%$$

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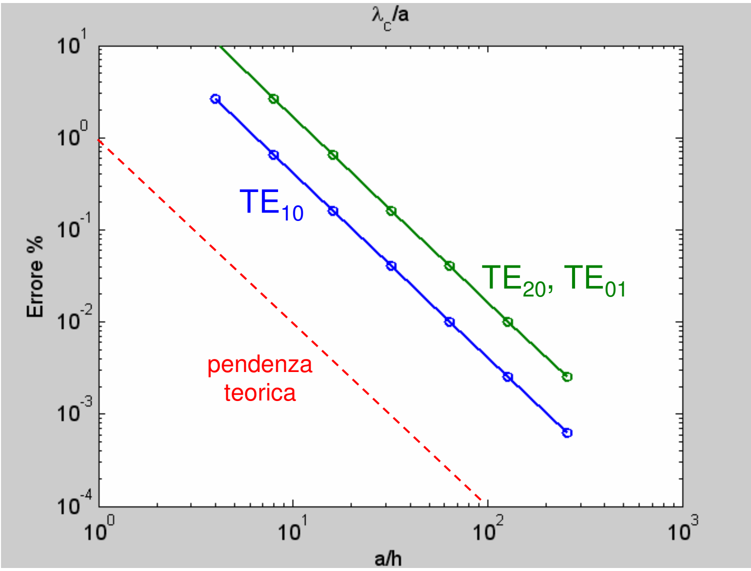
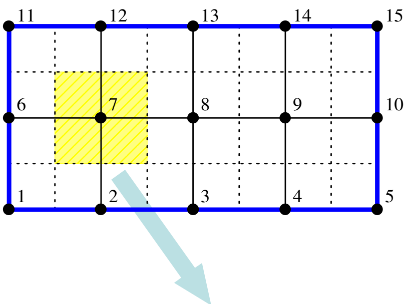
Andamento errore



Modi TM: sistema di equazioni



$$\begin{cases} \cancel{\Phi_2} + \cancel{\Phi_6} + \Phi_8 + \cancel{\Phi_{12}} - 4\Phi_7 = -k_c^{\,2} h^2 \Phi_7 \\ \cancel{\Phi_3} + \Phi_7 + \Phi_9 + \cancel{\Phi_{13}} - 4\Phi_8 = -k_c^{\,2} h^2 \Phi_8 \\ \cancel{\Phi_4} + \Phi_8 + \cancel{\Phi_{10}} + \cancel{\Phi_{14}} - 4\Phi_9 = -k_c^{\,2} h^2 \Phi_9 \end{cases}$$

Modi TM: equazione matriciale

$$\begin{cases} \Phi_8 - 4\Phi_7 = -k_c^{\,2} h^2 \Phi_7 \\ \Phi_7 + \Phi_9 - 4\Phi_8 = -k_c^{\,2} h^2 \Phi_8 \\ \Phi_8 - 4\Phi_9 = -k_c^{\,2} h^2 \Phi_9 \end{cases}$$

$$\mathbf{A\Phi} = \lambda \mathbf{B\Phi}$$

$$\begin{bmatrix} -4 & 1 & \circ \\ 1 & -4 & 1 \\ \circ & 1 & -4 \end{bmatrix} \begin{bmatrix} \Phi_7 \\ \Phi_8 \\ \Phi_9 \end{bmatrix} = -k_c^{\,2} h^2 \begin{bmatrix} 1 & \circ & \circ \\ \circ & 1 & \circ \\ \circ & \circ & 1 \end{bmatrix} \begin{bmatrix} \Phi_7 \\ \Phi_8 \\ \Phi_9 \end{bmatrix}$$

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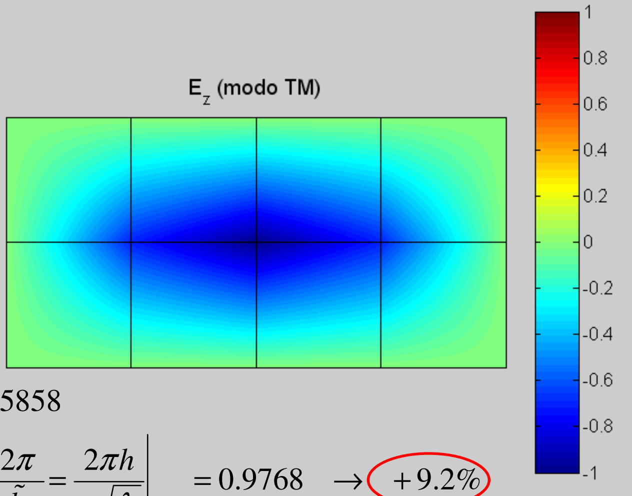
1° modo TM



$$\lambda_1 = 2.5858$$

$$\frac{\tilde{\lambda}_{c1}}{a} = \frac{2\pi}{a\tilde{k}_{c1}} = \left.\frac{2\pi h}{a\sqrt{\lambda_1}}\right|_{\frac{a}{h}=4} = 0.9768 \quad \rightarrow \quad +9.2\%$$

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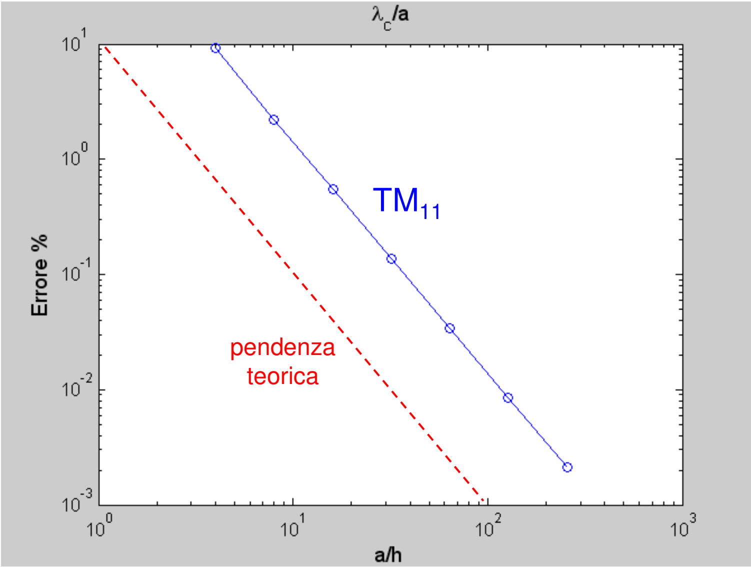
Andamento errore



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